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Non–orientable string one–loop corrections in the presence of a B field

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Abstract: We propose an expression for the string propagator on the boundary of the Möbius strip in the presence of a constant B field. We use it to compute the one–loop corrections to two–, three– and four–gluon amplitudes in an open string theory with orthogonal Chan–Paton factors. We show that these corrections in the field theory limit in 4D are compatible with the one–loop corrections of a renormalizable noncommutative $SO(N)$ gauge field theory.

Contents

1. Introduction

An interesting problem that has been raised in connection with the recent attention on noncommutative field theories as effective field theories of open strings attached to D–branes in the presence of a constant B field, is the existence of noncommutative gauge theories with gauge transformations valued in a Lie subalgebra of $u(N)$. There are several reasons why the existence of at least some of them is expected and desirable. From an abstract point of view, [1], there should not be any obstruction to constructing a noncommutative gauge field theory with any Lie algebra (even though this may not imply that these theories are effective field theories of the strings). On the other hand we know that noncommutative field theories retain certain features of string theory better than ordinary theories do, [2-31]. We have in mind here the ultraviolet convergence properties of noncommutative theories but, even more, the possibility of having soliton solutions in situations where ordinary theories are unfit to support them, [32]. This is particularly important in connection with tachyon condensation. In this regard, another important property is the possibility of embedding the Moyal product into the star product of open string field theory in a factorized way, [33]. It would be rather disappointing if such remarkable properties could not be extended, for example, to string theories or string field theories with orthogonal Chan-Paton factors.

Recently there have been a few attempts at defining and studying noncommutative versions of gauge field theories with orthogonal and symplectic, [34, 35], or even more general Lie algebras, [36]. These noncommutative theories have been defined at the semiclassical (tree) level. As soon as one tries to go beyond the tree level one has to face an undesired result: in four dimensions they look (at least naively) nonrenormalizable. One is tempted to dismiss this fact as a non–problem. After all, these are effective field theories, which are nonlocal as ordinary theories. However the right question we should ask is whether this corresponds to some feature (perhaps ill–definiteness) of the string theory the gauge field theory is supposed to represent in the low energy limit. To know the answer we have to study the one–loop corrections of the relevant string theory. This is what we want to do in this paper for an unoriented open string theory with orthogonal Chan–Paton factors in the presence of a background B field.

A D-brane with an $SO(N)$ (or $Sp(N)$) gauge theory on it can be found in correspondence with an orientifold: it corresponds to a set of branes and mirror branes which collapse on the orientifold. This fact entails a problem when we want to consider such a system in the presence of a B field. In fact the orientifold projection contains a space inversion which seems to exclude the presence of a B field in the final configuration. It was however argued in [34] that this is not a cogent difficulty, a way out can be found. Here we add an alternative simple argument to the one presented in [34]. In the original (before projection) theory one can always add to the B field a constant partwithout changing the equations of motion of (super)gravity. This constantpart is not directly affected by the string oscillators (which determine the equations of motion of the low energy effective action via the string amplitudes). On the other hand the orientifold projection operator is defined through the action on the string oscillators, so that a constant 'relic' B field may conceivably not be affected by the projection[1]. For similar considerations, see [37].

In this paper we give all this for granted and consider a set of D-branes collapsed over an orientifold with orthogonal (or symplectic) Chan–Paton factors in the presence of a constant B field. This is expected to give rise to a noncommutative $SO(N)$ (Sp(N)) gauge field theory. The tree level analysis of such theories has been carried out in [34]. As explained above, in this paper we wish to do the one–loop analysis. But this entails a new problem. In fact the sigma–model action for open strings attached to a D–brane is (we adopt the conventions of [38])

$$\frac{1}{4\pi\alpha'}\int_\Sigma d^2x\left(\sqrt{h}h^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j g_{ij} - 2\pi\alpha'\int_\Sigma d^2x\epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j B_{ij}\right) \tag{1.1}$$

where Σ is the string world–sheet, g_{ij} is the closed string metric and B_{ij} the components of the constant B field. At tree level the relevant world–sheet is the disk, while at one–loop the relevant world–sheets are the annulus and the Möbius band. Disk and annulus are orientable and the integrals in (1.1) are well–defined on such surfaces. But the Möbius strip in nonorientable and, while the first term in (1.1) is well defined on it, the second is not. The reason is that on nonorientable manifolds only *densities* can be integrated, see [39]. A density is an expression that, under a coordinate change, gets multiplied by the inverse modulus of the Jacobian of the partial derivatives (not just by the inverse Jacobian). Now, the first integrand in (1.1) is a density, while the second is not (it is the component of the pull–back of a two–form). Therefore the second part of (1.1) is meaningless when Σ is the Möbius band. However, since B is constant, in general we can replace (1.1) with

$$\frac{1}{4\pi\alpha'}\int_\Sigma d^2x\sqrt{h}h^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j g_{ij} - \frac{i}{2}\int_{\partial\Sigma} dt X^i\partial_t X^j B_{ij} \tag{1.2}$$

where ∂_t is the derivative tangent to the boundary $\partial\Sigma$. This expression is now well–defined also for the Möbius strip since its boundary (a circle) is orientable. From now on we will use (1.2) instead of (1.1). For convenience we denote henceforth a Möbius band by $\mathcal{M}$ and an annulus by $\mathcal{A}$.

[1] In $SO(N)$there is no global $U(1)$ factor as in $U(N)$. Therefore one may wonder whether the B field, which is not protected by the gauge invariant combination $B - dA$, might be gauged away. The answer is no, because the B field after the orientifold projection is not dynamical anymore, it does not appear in the effective action, so also its gauge properties disappear. Said differently, away from the orientifold every brane has a $U(1)$ field on it which guarantees the existence of a nonvanishing gauge invariant combination $B - dA$; it is natural to assume that in the collapsing limit, by continuity, the value of the gauge invariant combination $B - dA$ will be unchanged even though a (global) $U(1)$ A has disappeared.

After these preliminary specifications let us come to the content of this paper. In section 2 we compute the string propagator on $\mathcal{M}$ in the presence of B (the same propagator on the annulus can be found in the literature, [40, 20]). We actually find out that, as is to be expected from the above discussion, it is impossible to satisfy all the requirements of a true propagator on the full Möbius surface. However it is possible to extract a perfectly adequate expression for the propagator on the boundary of $\mathcal{M}$. This is enough for our purposes since we intend to compute amplitudes of vertex operators inserted exactly at the boundary of the world–sheet. Section 3 is devoted to the computation of 2–, 3– and 4–point gluon amplitudes and to their field theory limit in the absence of a B field. In section 4 we extend the same analysis to the case of nonzero B. *Our final result is that the noncommutative limiting field theory (in 4D) is one–loop renormalizable with the same renormalization constants as the corresponding ordinary $SO(N)$ gauge field theory.*

2. The string propagator on a non-orientable world-sheet

One–loop contributions in open unoriented string theory come from the annulus and the Möbius strip world–sheet. We will use the notation of [41]. The annulus will be represented either in the z–plane or in the ρ–plane. In the first case the annulus is represented in the most obvious way as the region $q \le |z| \le 1$, where q is the modulus. In the ρ-plane the annulus will be identified with the region $w \le |\rho| \le 1$ of the lower half plane with the lower and upper semicircle identified in such a way as to preserve the orientation of the surface (the two semicircles are 'parallel'). The map between the two representations is given by:

$$z = e^{2\pi i \frac{\ln\rho}{\ln w}}, \qquad \ln q = \frac{2\pi^2}{\ln w}$$

Alternatively the modulus is represented by the imaginary number $\tilde{\tau}$ defined by:

$$q = e^{i\pi\tilde{\tau}}, \qquad \tilde{\tau} = -\frac{2\pi i}{\ln w}$$

It is convenient to perform the modular transformation $\tilde{\tau} \to -1/\tilde{\tau}$. After this operation, following [42], we will parametrize the above variables as follows

$$w = e^{-2\tau}, \qquad \rho = e^{-2\nu} \tag{2.1}$$

where $\tau = -i\pi\tilde{\tau}$, for convenience.

The representation of the Möbius band is the same except that the upper semicircle in the ρ–plane is identified with the lower in an antiparallel way (see figure 1). The field theory limit corresponds to an infinitely thin annulus or band, i.e. $q \to 1$, which corresponds to $w \to 0$ or $\tau \to \infty$.

Our purpose in this paper is to compute amplitudes involving several gluon vertices inserted at the boundary of the annulus $\mathcal{A}$ or of the Möbius strip $\mathcal{M}$. To this end we need to know the string propagator on both surfaces. The string propagator in the annulus, in the presence of a B field, was calculated long ago in [40] and elaborated on in [20].

As for the string propagator on $\mathcal{M}$, in the absence of B, it can be found, for example, in [41]. Therefore what remains for us to do is to compute it in the presence of a constant B field. We will do it by finding the Green function with the appropriate boundary conditions on $\mathcal{M}$.

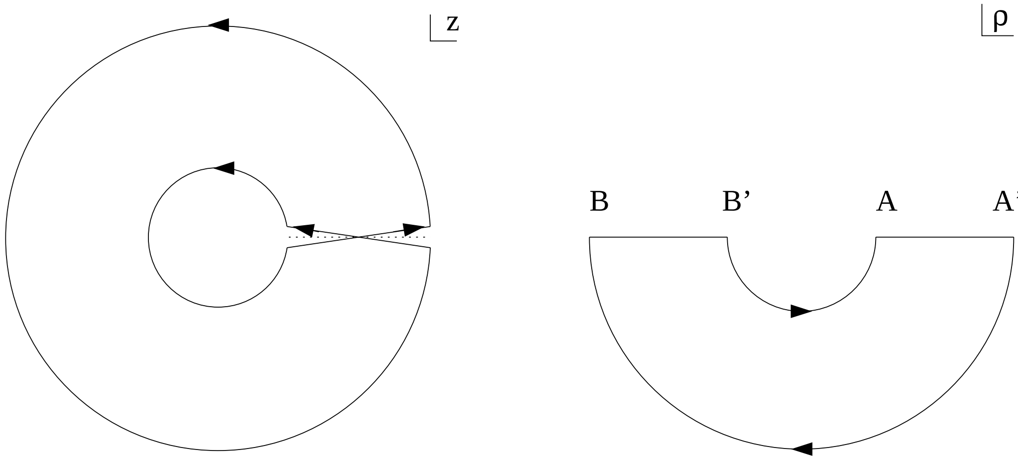


Figure 1: The representation of the Möbius band in the z and ρ planes

2.1 The Green function method

The problem is to find, on the surface Σ of interest (either $\mathcal{A}$ or $\mathcal{M}$), a solution (the Neumann function) of the equation

$$\nabla^2 \mathcal{G}^{ij}(x,x') = 2\pi\alpha'\delta^{ij}\delta(x-x') \tag{2.2}$$

satisfying the boundary conditions

$$\left(\partial_\perp \mathcal{G}^{ij}(x,x') - iF^i{}_k \partial_{||}\, \mathcal{G}^{kj}(x,x')\right)\Big|_{\partial\Sigma} = 0 \tag{2.3}$$

To simplify formulas we set henceforth $F = 2\pi\alpha' B$. Moreover we require that $\mathcal{G}^{ij}(x,x') = \mathcal{G}^{ji}(x',x)$. We recall that the above defining equations do not completely define the solution, see [40, 20].

In these equations x stands for either z or ρ. In the case $x = \rho$ the above equations become

$$\partial_\rho \partial_{\bar\rho} \mathcal{G}^{ij}(\rho,\rho') = \frac{\pi}{2}\alpha'\delta^{ij}\delta(\rho-\rho') \tag{2.4}$$

and

$$\left[(g+F)^i{}_k \rho\partial_\rho - (g-F)^i{}_k \bar\rho\partial_{\bar\rho}\right] \mathcal{G}^{kj}(\rho-\rho')\Big|_{\partial\Sigma} = 0 \tag{2.5}$$

The boundary $\partial\Sigma$ corresponds to real ρ with $w \leq |\rho| \leq 1$.

The solution we propose for the Möbius strip is as follows. Following [40, 20] we write it in the form

$$\frac{1}{\alpha'}\mathcal{G}^{ij}_{\mathcal{M}}(\rho,\rho') = g^{ij}\mathcal{I}^{\mathcal{M}}(\rho,\rho') + (2\hat{g}^{ij} - g^{ij})\mathcal{J}^{\mathcal{M}}(\rho,\rho') + \frac{\theta^{ij}}{\alpha'}\mathcal{K}^{\mathcal{M}}(\rho,\rho') \tag{2.6}$$

where

$$\hat{g}^{ij} = \left(\frac{1}{g+F}g\frac{1}{g-F}\right)^{ij}, \qquad \theta^{ij} = -2\pi\alpha'\left(\frac{1}{g+F}F\frac{1}{g-F}\right)^{ij} \tag{2.7}$$

are the open string metric and the deformation parameter, respectively, and

$$\begin{aligned}
\mathcal{I}_{\mathcal{M}}(\rho,\rho') &= \ln\left(\frac{|\tilde{\tau}|}{2}\right) + \frac{(\ln\frac{\rho}{\rho'})^2 + (\ln\frac{\bar{\rho}}{\bar{\rho}'})^2}{4\ln w} + \ln\left|\sqrt{\frac{\rho}{\rho'}} - \sqrt{\frac{\rho'}{\rho}}\right| \\
&\quad + \ln\prod_{n=1}^{\infty}\frac{|1-(-w)^n\frac{\rho}{\rho'}|\cdot|1-(-w)^n\frac{\rho'}{\rho}|}{(1-(-w)^n)^2} \qquad (2.8)\\
\mathcal{J}_{\mathcal{M}}(\rho,\rho') &= \ln\left(\frac{|\tilde{\tau}|}{2}\right) + \frac{(\ln\frac{\rho}{\bar{\rho}'})^2 + (\ln\frac{\bar{\rho}}{\rho'})^2}{4\ln w} + \ln\left|\sqrt{\frac{\rho}{\bar{\rho}'}} - \sqrt{\frac{\bar{\rho}'}{\rho}}\right| \\
&\quad + \ln\prod_{n=1}^{\infty}\frac{|1-(-w)^n\frac{\rho}{\bar{\rho}'}|\cdot|1-(-w)^n\frac{\bar{\rho}'}{\rho}|}{(1-(-w)^n)^2} \qquad (2.9)\\
\mathcal{K}_{\mathcal{M}}(\rho,\rho') &= \frac{(\ln\frac{\rho}{\bar{\rho}'})^2 - (\ln\frac{\bar{\rho}}{\rho'})^2}{4\ln w} + \ln\frac{\rho-\bar{\rho}'}{\bar{\rho}-\rho'} + \frac{1}{2}\ln\frac{\bar{\rho}\rho'}{\rho\bar{\rho}'} \\
&\quad + \ln\prod_{n=1}^{\infty}\frac{(1-(-w)^n\frac{\rho}{\bar{\rho}'})(1-(-w)^n\frac{\bar{\rho}'}{\rho})}{(1-(-w)^n\frac{\bar{\rho}}{\rho'})(1-(-w)^n\frac{\rho'}{\bar{\rho}})} \qquad (2.10)
\end{aligned}$$

There is a subtlety in the above definition: the log square term must be understood as

$$(\ln\frac{\rho}{\rho'})^2 = \frac{1}{4}(\ln(\frac{\rho}{\rho'})^2)^2$$

and so on.

Notice that $\mathcal{G}^{ij}_{\mathcal{M}}(\rho,\rho') = \mathcal{G}^{ji}_{\mathcal{M}}(\rho',\rho)$. It is now quite a standard matter to verify that eqs.(2.4) and (2.5) are satisfied. It is also easy to verify that the continuity condition on the boundary of the Möbius band is satisfied:

$$\mathcal{G}^{ij}_{\mathcal{M}}(1,\rho') = \mathcal{G}^{ij}_{\mathcal{M}}(-w,\rho'), \qquad \mathcal{G}^{ij}_{\mathcal{M}}(-1,\rho') = \mathcal{G}^{ij}_{\mathcal{M}}(w,\rho'), \qquad \forall\,\rho'$$

On the basis of the discussion in [40] the above would seem to contradict Gauss's theorem. We expect that the integral of the normal derivative around the boundary of a given worldsheet does not vanish but equals the integral of the right hand side of (2.2) over the worldsheet. In fact this contradiction does not exist. On the contrary this reveals an important fact about our propagator. If we compute the expression

$$[(g+F)_i{}^k\rho\partial_\rho - (g-F)_i{}^k\bar{\rho}\partial_{\bar{\rho}}]\mathcal{G}^{\mathcal{M}}_{kj}(\rho-\rho') \tag{2.11}$$

not only for ρ real (i.e. along the boundary of $\mathcal{M}$, where it vanishes) but also for $\rho = e^{-i\theta}$ and $\rho = -we^{i\theta}$ with $0 \leq \theta \leq \pi$, we find two very complicated expressions which, however, coincide up to a constant. So, upon integration of (2.11), we obtain a very simple result:

$$\int_0^\pi d\theta \left([(g+F)^i{}_k \rho\partial_\rho - (g-F)^i{}_k \bar\rho\partial_{\bar\rho}]\, \mathcal{G}^{kj}_{\mathcal{M}}(\rho-\rho')\Big|_{\rho=e^{-i\theta}} \right.$$
$$\left. + [(g+F)^i{}_k \rho\partial_\rho - (g-F)^i{}_k \bar\rho\partial_{\bar\rho}]\, \mathcal{G}^{kj}_{\mathcal{M}}(\rho-\rho')\Big|_{\rho=-we^{i\theta}} \right) = \pi\alpha'$$

i.e. Gauss's theorem is satisfied. This tells us that the normal derivative of our propagator $\mathcal{G}_{\mathcal{M}}$ has a line of discontinuity that cuts the Möbius strip open. Actually not only the normal derivative but the propagator itself is discontinuous across the identification line $\rho = e^{-i\theta}$, except at the two extreme points, which lie on the boundary of $\mathcal{M}$ [2]. By considering together with the propagator (2.6) its complex conjugate as the analytic continuation of the original one over the upper half plane, we can eliminate such a discontinuity. What we obtain in this way is a function defined on the double covering of the Möbius strip. The restriction to the boundary of the Möbius strip (real ρ) is however the same for the propagator and its complex conjugate.

Since we are interested in the propagator on the boundary of $\mathcal{M}$, we will take this restriction as the object we are looking for.

What remains for us to do is to extract its expression by taking the limit for ρ and ρ' approaching the real axis: $\mathcal{G}^{ij}_{\mathcal{M}} \to G^{ij}_{\mathcal{M}}$. We get

$$G^{ij}_{\mathcal{M}}(\rho,\rho') = 2\alpha'\hat{g}^{ij} G_{\mathcal{M}}(\rho,\rho') - \frac{i}{2}\theta^{ij}\epsilon(\rho-\rho') \tag{2.12}$$

where $G_{\mathcal{M}}(\rho,\rho')$ is given by

$$G^{+}_{\mathcal{M}}(\rho,\rho') = \ln\left(-\frac{\pi}{\ln w}\right) + \frac{(\ln\frac{\rho}{\rho'})^2}{2\ln w} + \ln\left|\sqrt{\frac{\rho}{\rho'}} - \sqrt{\frac{\rho'}{\rho}}\right|$$
$$+\ln\prod_{n=1}^{\infty} \frac{\left(1-(-w)^n\frac{\rho}{\rho'}\right)\left(1-(-w)^n\frac{\rho'}{\rho}\right)}{(1-(-w)^n)^2}, \quad \text{if} \quad \rho\rho' > 0 \tag{2.13}$$

$$G^{-}_{\mathcal{M}}(\rho,\rho') = \ln\left(-\frac{\pi}{\ln w}\right) + \frac{(\ln\left|\frac{\rho}{\rho'}\right|)^2}{2\ln w} + \ln\left|\sqrt{\left|\frac{\rho}{\rho'}\right|} + \sqrt{\left|\frac{\rho'}{\rho}\right|}\right|$$
$$+\ln\prod_{n=1}^{\infty} \frac{\left(1+(-w)^n\left|\frac{\rho}{\rho'}\right|\right)\left(1+(-w)^n\left|\frac{\rho'}{\rho}\right|\right)}{(1-(-w)^n)^2}, \quad \text{if} \quad \rho\rho' < 0 \tag{2.14}$$

This is the propagator we will use for our calculations in the following section[3].

[2] When $B = 0$, the propagator is given by the sum $g^{ij}(\mathcal{I}+\mathcal{J})$, which is continuous across the identification line $\rho = e^{-i\theta}$.

[3] Needless to say this leave open the problem of computing closed string states emission amplitudes in the present framework.

Finally we notice that by replacing $(-w)^n$ with w^n in (2.6) we get the Green function for the annulus, from which one can extract the planar and nonplanar propagators. This was done in [40] and in [20] and we will rely on those results.

To complete this section we write down the expression of the above Möbius propagator in the z plane. The latter is obtained from (2.6, 2.8, 2.9, 2.10), passing from ρ to z, changing $\tau \to -1/\tau$ and using well–known identities for the Jacobi theta–functions, [41]:

$$\frac{1}{\alpha'}\mathcal{G}^{ij}_{\mathcal{M}}(z,z') = g^{ij}\mathcal{I}^{\mathcal{M}}(z,z') + (2\hat{g}^{ij} - g^{ij})\mathcal{J}^{\mathcal{M}}(z,z') + \frac{\theta^{ij}}{\alpha'}\mathcal{K}^{\mathcal{M}}(z,z') \tag{2.15}$$

where

$$\mathcal{I}_{\mathcal{M}}(z,z') = \ln\left|\left(\frac{z}{z'}\right)^{\frac{1}{4}} - \left(\frac{z'}{z}\right)^{\frac{1}{4}}\right| + \ln\prod_{n=1}^{\infty}\frac{|1-(-\sqrt{q})^n\sqrt{\frac{z}{z'}}|\cdot|1-(-\sqrt{q})^n\sqrt{\frac{z'}{z}}|}{(1-(-\sqrt{q})^n)^2} \tag{2.16}$$

$$\mathcal{J}_{\mathcal{M}}(z,z') = \ln\left|(z\bar{z}')^{\frac{1}{4}} - (\bar{z}'z)^{-\frac{1}{4}}\right| + \ln\prod_{n=1}^{\infty}\frac{|1-(-\sqrt{q})^n\sqrt{z\bar{z}'}|\cdot|1-(-\sqrt{q})^n\frac{1}{\sqrt{\bar{z}'z}}|}{(1-(-\sqrt{q})^n)^2} \tag{2.17}$$

$$\mathcal{K}_{\mathcal{M}}(z,z') = \ln\frac{(z\bar{z}')^{\frac{1}{4}} - (z\bar{z}')^{-\frac{1}{4}}}{(z'\bar{z})^{\frac{1}{4}} - (z'\bar{z})^{-\frac{1}{4}}} + \ln\prod_{n=1}^{\infty}\frac{(1-(-\sqrt{q})^n\sqrt{z\bar{z}'})(1-(-\sqrt{q})^n\frac{1}{\sqrt{\bar{z}'z}})}{(1-(-\sqrt{q})^n\sqrt{\bar{z}z'})(1-(-\sqrt{q})^n\frac{1}{\sqrt{z'\bar{z}}})} \tag{2.18}$$

where $q = \exp[-\pi^2/\tau]$. Actually the expression for $\mathcal{K}_{\mathcal{M}}$ differs from (2.10) by a constant term, which is within the ambiguity remarked at the beginning this subsection. If $F = 0$ and we restrict the above expressions to the boundary, i.e. $|z| = |z'| = 1$, I becomes identical to J and the propagator reduces (up to an additive constant) to the expression one can find in [41].

3. Field theory limit of gluon amplitudes without B field

We wish to calculate string theory amplitudes and to extract from them information concerning the low energy effective field theory. In particular we are interested in the renormalization properties (in 4D) of the latter. For this reason in this section we intend to compute two–, three– and four–gluon one–loop amplitudes from string theory with $SO(N)$ CP factors and evaluate their field theory limit, more specifically the UV divergent contributions of the various amplitudes, in order to compare them with the field theory ones. While this has been done in detail in theories with unitary CP factors, [42], to our best knowledge nothing similar has been done for theories with orthogonal or symplectic CP factors. Therefore working out the field theory limit in the latter case without B field is a necessary preparation to the next section and a calculation interesting in itself. The novelty in this case is that, beside the annulus amplitudes, one has to consider also the Möbius strip ones.

The method we adopt here was developed over the years by several people, see [42] and references therein. It is based on calculations carried out in the framework of the bosonic string theory. Indeed it is enough to embed the gauge field theory we want to regularize in the bosonic string theory. It it is not even necessary that the string theory be critical.

As a regulator of a field theory a bosonic string theory in generic dimensions will do. For these and other considerations on the method used here, we refer to [42]).

We start by writing down the tree level gluon amplitudes with CP factors belonging to the Lie algebra $SO(N)$ at the lowest order in α'.

$$A^{(0)}(p_1,p_2) = 2i\mathrm{tr}(t^{a_1}t^{a_2})\epsilon_1\cdot\epsilon_2 p_1\cdot p_2 \tag{3.1}$$

$$A^{(0)}(p_1,p_2,p_3) = 4g_D\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3})\left(\epsilon_1\cdot\epsilon_2 p_2\cdot\epsilon_3+\epsilon_2\cdot\epsilon_3 p_3\cdot\epsilon_1+\epsilon_3\cdot\epsilon_1 p_1\cdot\epsilon_2\right) \tag{3.2}$$

$$\begin{aligned} A^{(0)}(p_1,p_2,p_3,p_4) = {}& 4ig_D^2\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\Bigg(\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\frac{p_1\cdot p_3}{p_1\cdot p_2} \\ &+\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4+\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3\frac{p_1\cdot p_3}{p_2\cdot p_3}\Bigg) \end{aligned} \tag{3.3}$$

To give a meaning to eq.(3.1) it is useful to introduce a small mass for the gluon: $p_i^2 = m^2$ (which is anyhow necessary as an IR cutoff, although we will not need it explicitly in the following). The above amplitudes have been normalized in such a way as to coincide with the corresponding tree level amplitudes in field theory. In particular, g_D is the D–dimensional gauge coupling, the t^a's are the generators of $SO(N)$ in the fundamental representation, the ϵ_i's are gluon polarizations and $p\cdot q = p_i\hat{g}^{ij}q_j$. Later on we will use the above formulas for $D=4$. In that case $g_D = g_4 \equiv g$. We recall that (3.3) contains, in field theory terms, also one–particle reducible contributions.

We write down now general form of the one–loop amplitudes (which, for later reference, is valid in general, also when a B field is switched on):

$$\begin{aligned} A^{(1)}(p_1,\ldots,p_M) = {}& \frac{1}{2}\chi_M\, f_N^{a_1,a_2,\ldots,a_M}\frac{g_D^M}{(4\pi)^{\frac{D}{2}}}(2\alpha')^{-\frac{D}{2}}\int\prod_{r=2}^{M}d\nu_r d\tau e^{2\tau}\tau^{-\frac{D}{2}} \\ &\times\prod_{n=1}^{\infty}\left(1-\eta_n e^{-2n\tau}\right)^{2-D}\exp\left[\sum_{r<s}p_r G(\nu_{rs})p_s\right] \\ &\times\exp\left[\sum_{r\neq s}\left(p_s\partial_r G(\nu_{sr})\epsilon_r+\frac{1}{2}\epsilon_r\partial_r\partial_s G(\nu_{sr})\epsilon_s\right)\right]_{\mathrm{m.l.}} \end{aligned}$$

where $\chi_M = i\,(1)$ for M even (odd). $f_N^{a_1,a_2,\ldots,a_M}$ is the group theory factor. It equals $N\mathrm{tr}(t^{a_1}\ldots t^{a_N})$ in the annulus case for planar amplitudes and $\mathrm{tr}(t^{a_1}\ldots t^{a_N})$ in the Möbius strip case. Moreover pGq stands for $p_iG^{ij}q_j$, $\nu_{rs} = \nu_r-\nu_s$ and $\partial_r = \frac{\partial}{\partial\nu_r}$. The factor $\eta_n = 1$ in the orientable case, $= (-1)^n$ in the non-orientable case.The suffix m.l. stands for multilinear, meaning that in the series expansion of the exponential we keep only the terms that are linear in each polarization. The propagator G is either the annulus or the Möbius strip propagator, and the integrals over the ν variables are evaluated in the appropriate regions of integration (moduli space).

The constants in front of the tree and one–loop amplitudes have been defined in such a way as to agree in the zero slope limit with the corresponding field theory results.

The strategy now consists in replacing in eq.(3.4) the appropriate propagators and singling out the regions of the moduli space which give rise to divergent contributions in the $\alpha' \to 0$ limit. This will be done explicitly below for the Möbius amplitudes. As for the annulus amplitudes, since their evaluation does not depend on the CP factors, we can borrow for them the analysis already carried out in [42] and [20, 22] in the case of unitary CP factors. These amplitudes split in general into planar and non–planar contributions. As for the latter we can rely on the results of [20], which, as expected, tells us that they do not give rise to UV divergences in the field theory limit[4]. The planar amplitudes do give rise to divergent contributions in the field theory limit. They have been analyzed in detail in [42]: assuming dimensional regularization in the $\alpha' \to 0$ limit, they reproduce exactly the results obtained in field theory with the background field method [43]. More precisely, one can single out the divergent part that corresponds in field theory to one–particle irreducible diagrams. The result can be written

$$A^{(1)}(p_1,...)\Big|_{\rm div} = -\frac{N}{2}\frac{g^2}{(4\pi)^2}\frac{11}{3}\frac{1}{\epsilon}A^{(0)}(p_1,...) \tag{3.4}$$

for two–, three– and four–point functions, with $\epsilon = 2 - D/2$. Throughout the paper the label *div* stands for irreducible divergent part, in the sense that in field theory these divergences correspond to one-particle irreducible diagrams. It is also possible to extract from string theory the one–loop one–particle reducible contributions but here we will not be concerned with them. In the remaining part of this section we will show how to extract relations similar to (3.4) for the Möbius amplitudes. Following [42], we will use two different methods. Since these two methods have already been carefully spelt out in [42] for the annulus amplitudes, we skip many details and focus on the peculiarities introduced by a non–orientable world–sheet.

3.1 Möbius amplitudes: first method

This method is based on the 'doubling trick', [41]. One can show that a large amount of information contained in a Möbius amplitude is captured by doubling the integration region. Let us start from the propagator along the boundary of $\mathcal{M}$, written as follows:

$$G_{\mathcal{M}}(\rho,\rho') = \ln\left[\frac{1-c}{\sqrt{c}}\exp\left(\frac{\ln^2 c}{2\ln w}\right)\prod_{n=1}^{\infty}\frac{(1-(-w)^n c)(1-(-w)^n/c)}{(1-(-w)^n)^2}\right],$$

where $c = \rho/\rho'$. This coincides with (2.13) provided $c \le 1$. Following [41], it can be recast in the form:

$$G_{\mathcal{M}}(\nu-\nu') = \ln\left[-\frac{4\pi}{\ln q}\sin\left(\frac{\pi(\nu-\nu')}{2}\right)\prod_{n=1}^{\infty}\frac{1-2(-\sqrt{q})^n\cos(\pi(\nu-\nu'))+q^n}{(1-2(-\sqrt{q})^n)^2}\right], \tag{3.5}$$

where $q = \exp[-\frac{\pi^2}{\tau}]$ and $\nu - \nu' = -\frac{1}{2}\ln c$. The form (3.5) of the Green function is periodic in the insertion coordinates ν's with a period double (4 instead of 2) with respect to the annulus case: this is because the boundary of the Möbius strip can be viewed as having

[4] A more careful statement is needed when a constant B field is present because of the UV/IR mixing. However in this paper we will not deal with this problem.

double length with respect to one of the two boundaries in the annulus. For our purposes we will need another form of $G_{\mathcal{M}}(\nu)$, first proposed by Fradkin and Tseytlin [44]. Using

$$\ln[1+b^2-2b\cos x] = -2\sum_{n=1}^{\infty}\frac{b^n}{n}\cos nx \tag{3.6}$$

and

$$\sum_{n=1}^{\infty} b^n = \frac{b}{1-b}, \tag{3.7}$$

we obtain

$$G_{\mathcal{M}}(\nu-\nu') = -\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{\pi n(\nu-\nu')}{\tau}\right)\left[\frac{1+(-\sqrt{q})^n}{1-(-\sqrt{q})^n}\right], \tag{3.8}$$

where we have used the regularization $\sum_{n=1}^{\infty} 1 = -\frac{1}{2}$ and we have neglected the terms that do not depend on ν. The effect of this regularization is that no negative powers of α' are generated in the integration over the variables ν's and τ, [42]. In this way we can replace the exponentials of the Green function simply by an infrared cutoff and extract from the amplitude only the terms proportional to $(\alpha')^{2-D/2}$. Keeping this fact in mind we rewrite the amplitude (3.4) as

$$A^{(1)}(p_1,\ldots,p_M) = \frac{1}{2}\chi_M f_N^{a_1,a_2,\ldots,a_M}\frac{g_D^M}{(4\pi)^{D/2}}(2\alpha')^{2-D/2}\int_0^{\infty}\mathcal{D}^{\mathcal{M}}\tau I_M^{(1)}(\tau) \tag{3.9}$$

where

$$I_{\mathcal{M}}^{(1)}(\tau) = (2\alpha')^{-2}\int_0^{\tau}d\nu_M\int_0^{\nu_M}d\nu_{M-1}\cdots\int_0^{\nu_3}d\nu_2 \tag{3.10}$$

$$\times\ \exp\left[\sum_{r<s}p_r G_{\mathcal{M}}(\nu_{rs})p_s\right]$$

$$\times\ \exp\left[\sum_{r\neq s}\left(p_r\partial_s G_{\mathcal{M}}(\nu_{sr})\epsilon_s+\frac{1}{2}\epsilon_r\partial_r\partial_s G_{\mathcal{M}}(\nu_{sr})\epsilon_s\right)\right]_{\text{m.l.}} \tag{3.11}$$

and

$$\mathcal{D}^{\mathcal{M}}\tau = d\tau w^{-1}\tau^{-D/2}\prod_{n=1}^{\infty}(1-(-w)^n)^{2-D} \tag{3.12}$$

Going to the variables $\hat{\nu} = \nu/\tau$ it is easier to implement the non–orientability of the Möbius band. We noticed above that the Green function $G^{\mathcal{M}}$ has double period in $\hat{\nu}$. The integration region must be chosen accordingly: the integration range is now $[0,2]$ instead of $[0,1]$, because we need to make two complete revolutions to go around the boundary back to the starting point.

$$I_{\mathcal{M}}^{(1)}(\tau) = (2\alpha')^{-2}\tau^{M-1}\int_0^2 d\hat{\nu}_M \int_0^{\hat{\nu}_M} d\hat{\nu}_{M-1}\cdots\int_0^{\hat{\nu}_3} d\hat{\nu}_2$$

$$\times \exp\left[\sum_{r<s} p_r G_{\mathcal{M}}(\nu_{rs}) p_s\right] \tag{3.13}$$

$$\times \exp\left[\sum_{r\neq s}\left(p_r\frac{1}{\tau}\hat{\partial}_s G_{\mathcal{M}}(\nu_{sr})\epsilon_s + \frac{1}{2}\frac{1}{\tau^2}\epsilon_r\hat{\partial}_r\hat{\partial}_s G_{\mathcal{M}}(\nu_{sr})\epsilon_s\right)\right] \tag{3.14}$$

For the two point function, after a partial integration with null boundary terms, we obtain

$$I_2^{(1)} = \epsilon_1\cdot\epsilon_2 p_1\cdot p_2\tau\int_0^2 d\hat{\nu}\left(\frac{1}{\tau}\hat{\partial}G_{\mathcal{M}}(\hat{\nu})\right)^2 e^{2\alpha' p_1\cdot p_2 G_{\mathcal{M}}(\hat{\nu})} \tag{3.15}$$

$$= \epsilon_1\cdot\epsilon_2 p_1\cdot p_2\int_0^2\left[\sum_{m=1}^{\infty}\frac{\pi}{\tau}\sin(\pi m\hat{\nu})\left[\frac{1+(-\sqrt{q})^m}{1-(-\sqrt{q})^m}\right]\right] \tag{3.16}$$

$$\times\left[\sum_{n=1}^{\infty}\frac{\pi}{\tau}\sin(\pi n\hat{\nu})\left[\frac{1+(-\sqrt{q})^n}{1-(-\sqrt{q})^n}\right]\right] \tag{3.17}$$

The partial integration has yielded the appropriate powers of α', so we can disregard the exponentials of the Green functions, and perform the $\hat{\nu}$ integration with the help of the formula

$$\int_0^2 dx\sin(\pi nx)\sin(\pi mx) = \delta_{nm} \tag{3.18}$$

and we are left with

$$I_2^{(1)} = \epsilon_1\cdot\epsilon_2 p_1\cdot p_2\frac{\pi^2}{\tau}\sum_{m=1}^{\infty}\left(\frac{1+(-\sqrt{q})^m}{1-(-\sqrt{q})^m}\right)^2 \tag{3.19}$$

Since the integration over τ will be shared by the 3– and 4–point functions, let us define

$$Z_{\mathcal{M}} = \pi^2\int_0^{\infty}\frac{\mathcal{D}^{\mathcal{M}}\tau}{\tau}\sum_{m=1}^{\infty}\left(\frac{1+(-\sqrt{q})^m}{1-(-\sqrt{q})^m}\right)^2 \tag{3.20}$$

The sum present in $Z_{\mathcal{M}}$ can be rewritten as

$$\sum_{n=1}^{\infty}\left(\frac{1+(-\sqrt{q})^m}{1-(-\sqrt{q})^m}\right)^2 = -4(-\sqrt{q})\frac{d}{d(-\sqrt{q})}\ln\left[(-\sqrt{q})^{1/8}\prod_{n=1}^{\infty}(1-(-\sqrt{q})^n)\right]\ , \tag{3.21}$$

then, using the relation (8.A.27) of [41], we can go to the k representation which is more suitable for the field theory limit

$$f(-\sqrt{q}) = \prod_{n=1}^{\infty}(1-(-\sqrt{q})^n) \tag{3.22}$$

$$= w^{1/24}q^{-1/48}\left(-\frac{\ln w}{\pi}\right)^{1/2} f(-w)$$

$$= w^{1/24}q^{-1/48}\left(-\frac{\ln w}{\pi}\right)^{1/2}\prod_{n=1}^{\infty}(1-(-w)^n).$$

and find the following expression for $Z_{\mathcal{M}}$

$$Z_{\mathcal{M}} = \pi^2 \int_0^\infty \frac{\mathcal{D}^{\mathcal{M}}\tau}{\tau} 4\sqrt{q}\frac{d}{d(-\sqrt{q})}\ln\left[(-\sqrt{q})^{1/8}\left(-\frac{\ln w}{\pi}\right)w^{1/24}q^{-1/48}f(-w)\right] \tag{3.23}$$

$$= 4\int_0^\infty \frac{\mathcal{D}^{\mathcal{M}}\tau}{\tau} w(\ln w)^2\left[-\frac{\pi^2}{12}\frac{1}{w(\ln k)^2} + \frac{1}{2w}\frac{1}{\ln w} + \frac{1}{24} + \sum_{n=1}^{\infty}\frac{n(-w)^{n-1}}{(1-(-w)^n)}\right] \tag{3.24}$$

Now we expand the partition function present in $\mathcal{D}^{\mathcal{M}}\tau$ in powers of $w = e^{-2\tau}$, and keep only the power $\tau^{1-D/2}$, that is the only one that gives rise to divergences in the dimensional regularization

$$Z_{\mathcal{M}} = \frac{2}{3}(26-D)\int_0^\infty d\tau\tau^{1-D/2}e^{-2\alpha' m^2\tau} = \frac{2}{3}(26-D)\Gamma\left(2-\frac{D}{2}\right)(2\alpha' m^2)^{D/2-2} \tag{3.25}$$

After setting $\epsilon = 2 - D/2$, we obtain for the two point function

$$\begin{aligned} A^{(1)}_{\mathcal{M}}(p_1,p_2)\Big|_{\text{div}} &= \frac{i}{2}\text{tr}(t^{a_1}t^{a_2})\frac{g_D^2}{(4\pi)^{D/2}}(2\alpha')^{2-D/2}(2\alpha' m^2)^{D/2-2} \\ &\times \epsilon_1\cdot\epsilon_2 p_1\cdot p_2\frac{2}{3}(26-D)\Gamma\left(2-\frac{D}{2}\right) \\ &= \frac{g^2}{(4\pi)^2}\frac{11}{3}\frac{1}{\epsilon}A^{(0)}(p_1,p_2) \end{aligned} \tag{3.26}$$

where $g \equiv g_4$. For three gluons we have

$$I_3^{\mathcal{M}} = \frac{1}{\tau}\int_0^2 d\hat{\nu}_3\int_0^{\hat{\nu_3}} d\hat{\nu_2}\left\{-\epsilon_1\cdot\epsilon_2\hat{\partial}_2^2 G(\nu_2)\left[p_1\cdot\epsilon_3\hat{\partial}_3 G(\nu_3) + p_2\cdot\epsilon_3\hat{\partial}_3 G(\nu_{32})\right] + \ldots\right\}$$

where the dots stand for the terms obtained by cyclic symmetry and for terms of higher order in α'. The power of α' in the expression above is the correct one, without partial integration: also in this case we can neglect the exponentials, because they are irrelevant for ultraviolet divergencies. The integral over ν's coordinates is done again using the formula (3.18):

$$I_3^{\mathcal{M}} = 2\Big(\epsilon_1\cdot\epsilon_2 p_2\cdot\epsilon_3 + \epsilon_2\cdot\epsilon_3 p_3\cdot\epsilon_1 + \epsilon_1\cdot\epsilon_3 p_1\cdot\epsilon_2\Big)\frac{\pi^2}{\tau}\sum_{n=1}^{\infty}\left(\frac{1-(-\sqrt{q})^n}{(1-(-\sqrt{q})^n)}\right)^2 \tag{3.27}$$

$$\begin{aligned} A^{(1)}_{\mathcal{M}}(p_1,p_2,p_3)\Big|_{\text{div}} &= \frac{1}{2}\text{tr}(t^{a_1}t^{a_2}t^{a_3})\frac{g_D^2}{(4\pi)^{D/2}}(2\alpha')^{2-D/2}(2\alpha' m^2)^{D/2-2} \\ &\times 2\Big(\epsilon_1\cdot\epsilon_2 p_2\cdot\epsilon_3 + \epsilon_2\cdot\epsilon_3 p_3\cdot\epsilon_1 + \epsilon_1\cdot\epsilon_3 p_1\cdot\epsilon_2\Big)\frac{2}{3}(26-D)\Gamma\left(2-\frac{D}{2}\right) \\ &= \frac{g^2}{(4\pi)^2}\frac{11}{3}\frac{1}{\epsilon}A^{(0)}(p_1,p_2,p_3) \end{aligned} \tag{3.28}$$

Finally for four gluons we have

$$\begin{aligned} I_4^{\mathcal{M}} &= \frac{1}{\tau^2}\int_0^2 d\hat{\nu}_4\int_0^{\hat{\nu}_4} d\hat{\nu}_3\int_0^{\hat{\nu}_3} d\hat{\nu}_2\Big(\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\hat{\partial}_2^2 G(\nu_2)\hat{\partial}_4^2 G(\nu_{43}) \\ &+ \epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4\hat{\partial}_3^2 G(\nu_3)\hat{\partial}_4^2 G(\nu_{42}) \\ &+ \epsilon_1\cdot\epsilon_4\epsilon_3\cdot\epsilon_2\hat{\partial}_4^2 G(\nu_4)\hat{\partial}_3^2 G(\nu_{32}) + \ldots\Big) \end{aligned} \tag{3.29}$$

Again we have the correct power of α' without partial integration and we can discard the exponential; the dots denotes terms proportional to the external momenta that will play no role because they are not present in the 1PI tree level diagrams.

$$I_4^{\mathcal{M}} = 2\left(-\frac{1}{2}\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 + \epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 - \frac{1}{2}\epsilon_1\cdot\epsilon_4\epsilon_3\cdot\epsilon_2\right)\frac{\pi^2}{\tau}\sum_{n=1}^{\infty}\left(\frac{1+(-\sqrt{q})^n}{1-(-\sqrt{q})^n}\right)^2$$

Therefore

$$\begin{aligned} A^{(1)}_{\mathcal{M}}(p_1,p_2,p_3,p_4)\Big|_{\rm div} &= \frac{i}{2}\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\frac{g_D^2}{(4\pi)^{D/2}}(2\alpha')^{2-D/2}(2\alpha' m^2)^{D/2-2} \qquad (3.30)\\ &\times 2\left(-\frac{1}{2}\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 + \epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 - \frac{1}{2}\epsilon_1\cdot\epsilon_4\epsilon_3\cdot\epsilon_2\right)\frac{2}{3}(26-D)\Gamma\left(2-\frac{D}{2}\right)\\ &= \frac{g^2}{(4\pi)^2}\frac{11}{3}\frac{1}{\epsilon}A^{(0)}(p_1,p_2,p_3,p_4)\end{aligned}$$

We can now summarize our results by collecting together the planar amplitudes (3.4) and the Möbius ones. The final result is

$$A^{(1)}(p_1,...)\Big|_{\rm div} = -\frac{N-2}{2}\frac{g^2}{(4\pi)^2}\frac{11}{3}\frac{1}{\epsilon}A^{(0)}(p_1,...) \qquad (3.31)$$

3.2 Möbius amplitudes: second method

The second method is more laborious, but it has the advantage that one can single out more explicitly the regions of the moduli space corresponding to the different divergent contributions and thus provides a better understanding of the field theory limit. The $\alpha' \to 0$ limit corresponds to the parameters τ and ν_r going to infinity, or, more precisely, to $\tau\to\infty$ and $\hat{\nu}_r = \frac{\nu_r}{\tau}$ finite. Therefore, on the basis of (3.4), we need the corresponding asymptotic expansion of $G_{\mathcal{M}}(\nu)$ and its derivatives. The latter is given, up to $\mathcal{O}(e^{-4\tau})$ terms, by (from now on we drop the subscript $_{\mathcal{M}}$ from the propagator):

$$\begin{aligned} G^+(\nu) &= -\hat{\nu}^2\tau + \hat{\nu}\tau - e^{-2\hat{\nu}\tau} + e^{-2\tau}\left(e^{-2\hat{\nu}\tau} + e^{2\hat{\nu}\tau} - 1\right)\\ \partial_\nu G^+(\nu) &= -2\hat{\nu} + 1 + 2e^{-2\hat{\nu}\tau} + 2e^{-2\tau}\left(e^{2\hat{\nu}\tau} - e^{-2\hat{\nu}\tau}\right) \qquad (3.32)\\ \partial_\nu^2 G^+(\nu) &= -\frac{2}{\tau} - 4e^{-2\hat{\nu}\tau} + 4e^{-2\tau}\left(e^{2\hat{\nu}\tau} + e^{-2\hat{\nu}\tau}\right)\end{aligned}$$

and

$$\begin{aligned} G^-(\nu) &= -\hat{\nu}^2\tau + \hat{\nu}\tau + e^{-2\hat{\nu}\tau} - e^{-2\tau}\left(e^{-2\hat{\nu}\tau} + e^{2\hat{\nu}\tau} + 1\right)\\ \partial_\nu G^-(\nu) &= -2\hat{\nu} + 1 - 2e^{-2\hat{\nu}\tau} - 2e^{-2\tau}\left(e^{2\hat{\nu}\tau} - e^{-2\hat{\nu}\tau}\right) \qquad (3.33)\\ \partial_\nu^2 G^-(\nu) &= -\frac{2}{\tau} - 4e^{-2\hat{\nu}\tau} - 4e^{-2\tau}\left(e^{2\hat{\nu}\tau} + e^{-2\hat{\nu}\tau}\right)\end{aligned}$$

To compute the one–loop amplitude we have to specify which partial propagator G^+ or G^- we have to insert in eq.(3.4). To this end we split the boundary of $\mathcal{M}$ into two parts AA' lying in the positive real ρ axis, and BB' along the negative ρ axis (see figure). One has to consider all the configurations which are compatible with any given ordering of the gluon insertions along the boundary of $\mathcal{M}$.

3.2.1 Two–gluon amplitude

In the two–gluon amplitude only one propagator is involved. Therefore the two–gluon amplitude on $\mathcal{M}$ contains two contributions, one with G^+ corresponding to the gluon insertions in the same interval AA' or BB' and the other with G^- corresponding to one insertion in AA' and the other in BB'. We will use translational invariance in order to fix the insertion 1 at the point A', i.e. $\rho_1 = 1$ or $\nu_1 = 0$. After these preliminaries we insert all the data in eq.(3.4) and find

$$A^{(1)}_{\mathcal{M}}(p_1,p_2) = \frac{i}{2}\mathrm{tr}(t^{a_1}t^{a_2})\frac{g_D^2}{(4\pi)^{\frac{D}{2}}}(2\alpha')^{2-\frac{D}{2}}\int_0^\infty d\tau e^{2\tau}\tau^{-\frac{D}{2}}\prod_{n=1}^{\infty}\left(1-(-1)^n e^{-2n\tau}\right)^{2-D}$$
$$\times\,(-\epsilon_1\cdot\epsilon_2)\int_0^\tau d\nu\left(e^{2\alpha' p_1\cdot p_2 G^+(\nu)}\partial_\nu^2 G^+(\nu) + e^{2\alpha' p_1\cdot p_2 G^-(\nu)}\partial_\nu^2 G^-(\nu)\right) \quad (3.34)$$

where $\nu = \nu_2$.

Now we integrate by parts in ν and disregard the contributions at $\nu = 0, \tau$, since, as was noticed in [42], they correspond in field theory to massless tadpole contributions, which are defined to vanish in dimensional regularization. Therefore the RHS of (3.34) can be replaced by:

$$\frac{i}{2}\mathrm{tr}(t^{a_1}t^{a_2})\frac{g_D^2}{(4\pi)^{\frac{D}{2}}}(2\alpha')^{2-\frac{D}{2}}\int_0^\infty d\tau e^{2\tau}\tau^{-\frac{D}{2}}\prod_{n=1}^{\infty}\left(1-e^{-2n\tau}\right)^{2-D}$$
$$(\epsilon_1\cdot\epsilon_2\, p_1\cdot p_2)\int_0^\tau d\nu\left(e^{2\alpha' p_1\cdot p_2 G^+(\nu)}\left(\partial_\nu G^+(\nu)\right)^2 + e^{2\alpha' p_1\cdot p_2 G^-(\nu)}\left(\partial_\nu G^-(\nu)\right)^2\right) \quad (3.35)$$

At this point we insert the expansions (3.32) and (3.33) and evaluate the ν integral first. One notices that the two exponentials $e^{2\alpha' p_1\cdot p_2 G^{\pm}(\nu)}$ for large τ can be written as $e^{2\alpha' p_1\cdot p_2\,(\hat{\nu}-\hat{\nu}^2)\tau}$ and play the role of a cutoff factor. Therefore, for large τ, the ν integral in (3.35) is determined by

$$\int_0^\tau d\nu\, 2\,\left((1-2\hat{\nu})^2 + 8e^{-2\tau}\right)e^{2\alpha' p_1\cdot p_2\,(\hat{\nu}-\hat{\nu}^2)\tau}$$

Now inserting this equation back into (3.35), we see that there are contributions to the τ integral proportional to $e^{2\tau}$. These are recognized to be contributions from the tachyon and must be discarded by hand (this ad hoc operation is the price we have to pay for having embedded our gauge theory in the bosonic string rather than in a superstring theory). The terms of zeroth order in $e^{2\tau}$ are the relevant ones for our purposes. As shown in [42], these integrals can be exactly evaluated and the pole in $\epsilon = \frac{4-D}{2}$ easily extracted. The result is

$$A^{(1)}_{\mathcal{M}}(p_1,p_2)\Big|_{\mathrm{div}} = i\mathrm{tr}(t^{a_1}t^{a_2})\frac{g^2}{(4\pi)^2}\epsilon_1\cdot\epsilon_2\, p_1\cdot p_2\cdot\frac{11}{3}\frac{1}{\epsilon} \quad (3.36)$$

which, if we forget the factor of N, is twice the planar contribution with opposite sign. If we put together the results for the planar annulus amplitude and the Möbius strip we finally obtain for the 1PI divergent part of the two–gluon amplitude

$$A^{(1)}(p_1,p_2)\Big|_{\mathrm{div}} = -\frac{N-2}{2}\frac{g^2}{(4\pi)^2}\frac{11}{3}\frac{1}{\epsilon}A^{(0)}(p_1,p_2) \quad (3.37)$$

This is exactly what is expected from renormalization theory in the background field method formalism.

3.2.2 Three–gluon amplitude

The three–gluon amplitude involves two propagators and four possible configurations for any given ordering of the external legs, [45]. The four configurations can be classified as follows. The orientation of the boundary of $\mathcal{M}$ is chosen from A' to A and from B to B'. We call it the standard orientation. We consider the three insertions at ρ_1, ρ_2, ρ_3 ordered according to the standard orientation and set $\rho_1 = A'$, see figure. Now we append by convention a $+$ or a $-$ to ρ according to whether ρ falls in the interval AA' or in BB'. The four configurations are then specified as follows

- s1: $(\rho_1^+, \rho_2^+, \rho_3^+)$
- s2: $(\rho_1^+, \rho_2^+, \rho_3^-)$
- s3: $(\rho_1^+, \rho_2^-, \rho_3^-)$
- s4: $(\rho_1^+, \rho_3^-, \rho_2^+)$

Each triple is in order of decreasing modulus. For instance, s4 means $|\rho_1| \geq |\rho_3| \geq |\rho_2|$ and that ρ_1 and ρ_2 are in AA' while ρ_3 is in BB'. s1–s4 specify distinct sectors of the integration region (moduli space).

The amplitude given by (3.4) contains three pieces, which are proportional to the three terms contained in the RHS of (3.2). We will consider here the one proportional to $\epsilon_1 \cdot \epsilon_2\, p_2 \cdot \epsilon_3$. The corresponding coefficient in $A^{(1)}(p_1, p_2, p_3)$ is given by

$$\frac{1}{2}\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3})\frac{g_D^3}{(4\pi)^{\frac{D}{2}}}(2\alpha')^{\frac{4-D}{2}} \int_0^\infty d\tau e^{2\tau}\tau^{-\frac{D}{2}}\prod_{n=1}^{\infty}\left(1-(-1)^n e^{-2n\tau}\right)^{2-D} \tag{3.38}$$
$$\Big\{\int_0^\tau d\nu_3 \int_0^{\nu_3} d\nu_2\, e^{2\alpha'[p_1\cdot p_2 G_+(\nu_2)+p_2\cdot p_3 G_+(\nu_{32})+p_1\cdot p_3 G_+(\nu_3)]}\partial^2_{\nu_2}G_+(\nu_2)\partial_{\nu_3}\left(G_+(\nu_{32})-G_+(\nu_3)\right)$$
$$+\int_0^\tau d\nu_3 \int_0^{\nu_3} d\nu_2\, e^{2\alpha'[p_1\cdot p_2 G_+(\nu_2)+p_2\cdot p_3 G_-(\nu_{32})+p_1\cdot p_3 G_-(\nu_3)]}\partial^2_{\nu_2}G_+(\nu_2)\partial_{\nu_3}\left(G_-(\nu_{32})-G_-(\nu_3)\right)$$
$$+\int_0^\tau d\nu_3 \int_0^{\nu_3} d\nu_2\, e^{2\alpha'[p_1\cdot p_2 G_-(\nu_2)+p_2\cdot p_3 G_+(\nu_{32})+p_1\cdot p_3 G_-(\nu_3)]}\partial^2_{\nu_2}G_-(\nu_2)\partial_{\nu_3}\left(G_+(\nu_{32})-G_-(\nu_3)\right)$$
$$+\int_0^\tau d\nu_2 \int_0^{\nu_2} d\nu_3\, e^{2\alpha'[p_1\cdot p_2 G_+(\nu_2)+p_2\cdot p_3 G_-(\nu_{32})+p_1\cdot p_3 G_-(\nu_3)]}\partial^2_{\nu_2}G_+(\nu_2)\partial_{\nu_3}\left(G_-(\nu_{32})-G_-(\nu_3)\right)\Big\}$$

The last four lines in this equation correspond to the contributions from the four configurations listed above, in the same order. As analyzed in [42], the divergent contributions corresponding to 1PI diagrams in the $\alpha' \to 0$ limit, come from two different regions of the moduli space, which we call type I and type II.

The type I region corresponds to the three insertion points being kept widely separated while $\tau \to \infty$, i.e. while the Möbius strip shrinks to zero size ($w \to 0, q \to 1$). Intuitively, this corresponds in the field theory language to Feynman diagrams with three propagators and three three–point vertices. This means that ν_3 and ν_{32} are of order τ while $\tau \to \infty$.

It is possible to show that these contributions come only from the first terms (those not containing exponentials) in the asymptotic expansions (3.32, 3.33). We seem to have four contributions of this type, corresponding to the four configurations s1–s4. However this is not the case. Only two of them contribute to type I, precisely s1 and s3. In s2 and s4, point 2 and point 3 are bound to lie on opposite sides of the band; in the field theory limit these contributions do not flow toward the expected Feynman diagrams. In a sense they are analogous to the nonplanar ones.

To evaluate the type I contribution we remark that the exponentials in eq.(3.38) play simply the role of dumping factors. Therefore we simplify things by replacing them with a universal dumping factor $e^{-2\alpha' m^2 \tau}$. After discarding the tachyon contribution one can see that the relevant UV divergent part from region of type I in eq.(3.38) is contained in

$$\frac{1}{2}\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3})\frac{g_D^3}{(4\pi)^{\frac{D}{2}}}(2\alpha')^{\frac{4-D}{2}}\int_0^\infty d\tau\tau^{-\frac{D}{2}}(2-D)\int_0^\tau d\nu_3\int_0^{\nu_3} d\nu_2\, e^{-2\alpha' m^2\tau} 8\left(\frac{\nu_2}{\tau^2}\right)$$

After a standard integration, this becomes

$$-\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3})\frac{g_D^3}{(4\pi)^{\frac{D}{2}}}\frac{4}{3}m^{D-4}\Gamma(\epsilon)$$

Collecting the above results and setting $D = 4$ one finds the type I contribution to the divergent part of the three–gluon amplitude is:

$$A^{(1)}_{\mathcal{M}}(p_1,p_2,p_3)\Big|_I = -\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3})\,\frac{g^3}{(4\pi)^2}\,\epsilon_1\cdot\epsilon_2\, p_2\cdot\epsilon_3\,\frac{4}{3}\,\frac{1}{\epsilon} \tag{3.39}$$

Let us pass now to the type II region. It is the region in the moduli space where two insertion points come close together like $1/\tau$ as $\tau\to\infty$. In field theory such terms correspond to one–loop three–gluon diagrams with one four–point vertex. There are three possibilities: either $\rho_1\to 1$, or $\rho_1\to\rho_2$, or $\rho_3\to -w$. These correspond to either $\hat\nu_2\sim\mathcal{O}(\tau^{-1})$ or $\hat\nu_{32}\sim\mathcal{O}(\tau^{-1})$ or $\hat\nu_2\sim\mathcal{O}(\tau^{-1})$. In field theory terms this corresponds to Feynman diagram with two internal propagators and one four–point vertex.

Using the asymptotic expansions (3.32, 3.33) into (3.38) one can see that the type II contributions can only come from the exponential terms in (3.32, 3.33). Once again, however, we should not apply the formulas mechanically. The type II contributions of the sectors s1–s4 must be carefully evaluated. For instance it is evident that in s4 the punctures 2 and 3 cannot approach each other because they are confined to lie on opposite sides of the band. On the other hand ρ_3 cannot go to $-w$ because $|\rho_3|\geq|\rho_2|$, and, for the same reason ρ_2 cannot go to 1. Therefore neither 3 nor 2 can get close to 1. Thus sector s4 is not going to contribute to type II. On the other hand, in s1 we have the possible collapses $2\to 1$ and $2\to 3$, in s2 we have the only possible collapse $3\to 1$, while in s4 we can have both $3\to 1$ and $2\to 3$. As it turns out, $2\to 1$ does not contributes to the divergent part. Carrying out the explicit calculations, the divergent part of (3.38), as far as type II is concerned, is contained in

$$\frac{1}{2}\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}) \quad \frac{g_D^3}{(4\pi)^{\frac{D}{2}}}(2\alpha')^{\frac{4-D}{2}} \int_0^\infty d\tau \tau^{-\frac{D}{2}} e^{2\tau} \int_0^\tau d\nu_3 \int_0^{\nu_3} d\nu_2 \, e^{-2\alpha' m^2 \tau}$$
$$\Big[8e^{-2\tau+2(\nu_3-\nu_2)} - 8^{-2\tau+2(\nu_2-\nu_3)}$$
$$+ 8e^{-2\nu_3} - 8e^{-4\tau+2\nu_3}$$
$$+ 8e^{-2\tau+2(\nu_3-\nu_2)} - 8^{-2\tau+2(\nu_2-\nu_3)} + 8e^{-2\nu_3} - 8e^{-4\tau+2\nu_3} \Big] \tag{3.40}$$

where the last three lines correspond to the s1, s2 and s3 contributions, respectively. The calculation now is straightforward. Setting $D = 4-2\epsilon$ one finds that the type II contribution to the divergent part of the three–gluon amplitude is:

$$A^{(1)}_{\mathcal{M}}(p_1, p_2, p_3)\Big|_{II} = \mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}) \, \frac{g^3}{(4\pi)^2} \, \epsilon_1 \cdot \epsilon_2 \, p_2 \cdot \epsilon_3 \, 16 \, \frac{1}{\epsilon} \tag{3.41}$$

Finally the total divergent part for the three–gluon amplitude is

$$A^{(1)}_{\mathcal{M}}(p_1, p_2, p_3)\Big|_{\mathrm{I+II}} = \mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}) \, \frac{g^3}{(4\pi)^2} \, \epsilon_1 \cdot \epsilon_2 \, p_2 \cdot \epsilon_3 \, \cdot \frac{44}{3} \, \frac{1}{\epsilon} \tag{3.42}$$

Therefore

$$A^{(1)}(p_1, p_2, p_3)\Big|_{\mathrm{I+II}} = -\frac{N-2}{2} \frac{g^2}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} A^{(0)}(p_1, p_2, p_3) \tag{3.43}$$

3.2.3 Four–gluon amplitude

The four–gluon amplitude involves three propagators and eight possible configurations for any given ordering of the external legs, see [41]. The eight configurations can be classified as above. We consider the four insertions at $\rho_1, \rho_2, \rho_3, \rho_4$ ordered according to the standard orientation of the boundary of $\mathcal{M}$ and set $\rho_1 = A'$. The corresponding eight sectors of integration are then specified as follows

- s1: $(\rho_1^+, \rho_2^+, \rho_3^+, \rho_4^+)$
- s2: $(\rho_1^+, \rho_2^+, \rho_3^+, \rho_4^-)$
- s3: $(\rho_1^+, \rho_2^+, \rho_3^-, \rho_4^-)$
- s4: $(\rho_1^+, \rho_2^-, \rho_3^-, \rho_4^-)$
- s5: $(\rho_1^+, \rho_4^-, \rho_2^+, \rho_3^-)$
- s6: $(\rho_1^+, \rho_4^-, \rho_2^+, \rho_3^+)$
- s7: $(\rho_1^+, \rho_2^+, \rho_4^-, \rho_3^-)$
- s8: $(\rho_1^+, \rho_3^-, \rho_4^-, \rho_2^+)$

Each quadruple is written in order of decreasing modulus.

Now we single out in (3.4) the piece proportional to $\epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot \epsilon_4$ (the other two pieces can be dealt with similarly, see [42]) and simplify the resulting expression as in the three–gluon case. In particular we replace the exponential factors with a unique dumping factor $e^{-2\alpha' m^2 \tau}$.

Next we discus the contributions from region I and II. To this end we avoid explicitly writing down encumbering equations. Let us recall that type I contributions come from well separated configurations of the punctures in the limit $\tau \to \infty$, they correspond in field theory to Feynman diagram with four internal propagators. The only two sectors that can contribute are s1 and s4. All the other sectors are non–planar–like in that they contain at least two points on opposite sides of the band. Their field theory limit is different from that expected for type I contributions.

As for type II contributions they correspond to two separate couples of points coming simultaneously together like $\mathcal{O}(1/\tau)$ as $\tau \to \infty$. In field theory this correspond to Feynman diagram with two internal propagators. Sector by sector we find: in s1 we can have $2 \to 1$ and $3 \to 4$; in s2 we can have $2 \to 3$ and $\rho_4 \to -w$, i.e. $4 \to 1$; in s3 we can have $2 \to 1$ and $3 \to 4$; in s4 we can have $2 \to 3$ and $\rho_4 \to -w$, i.e. $4 \to 1$; no two separate couples of points can come simultaneously together in the remaining sectors. So sectors s5–s8 do not contribute neither to type I nor to type II divergences.

Now, going to explicit formulas, we find that the relevant multiplicative factor of $\epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot \epsilon_4$ in $A^{(1)}(p_1, p_2, p_3, p_4)$ is

$$
\begin{aligned}
&\frac{i}{2}\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\frac{g_D^4}{(4\pi)^{\frac{D}{2}}}(2\alpha')^{\frac{4-D}{2}} \int_0^\infty d\tau e^{2\tau}\tau^{-\frac{D}{2}}\prod_{n=1}^{\infty}\left(1-(-1)^n e^{-2n\tau}\right)^{2-D} \\
&\int_0^\tau d\nu_4 \int_0^{\nu_4} d\nu_3 \int_0^{\nu_3} d\nu_2\, e^{-2\alpha' m^2 \tau}\Big[\partial^2_{\nu_3}G_+(\nu_3)\partial^2_{\nu_4}G_+(\nu_{42}) \\
&+\partial^2_{\nu_3}G_+(\nu_3)\partial^2_{\nu_4}G_-(\nu_{42}) + \partial^2_{\nu_3}G_-(\nu_3)\partial^2_{\nu_4}G_-(\nu_{42}) + \partial^2_{\nu_3}G_-(\nu_3)\partial^2_{\nu_4}G_+(\nu_{42})\Big] \quad (3.44)
\end{aligned}
$$

where the terms in square brackets refer to sector s1 down to s4, respectively.

It remains for us to evaluate the above integral for type I and II. As pointed out above the type I contributions come only from the first terms (those not containing exponentials) in the asymptotic expansions (3.32, 3.33).

$$
\frac{i}{2}\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\frac{g_D^4}{(4\pi)^{\frac{D}{2}}}(2\alpha')^{\frac{4-D}{2}} \int_0^\infty d\tau \tau^{-\frac{D}{2}}(2-D)\int_0^\tau d\nu_4 \int_0^{\nu_4} d\nu_3 \int_0^{\nu_3} d\nu_2\, e^{-2\alpha' m^2 \tau}\left(\frac{8}{\tau^2}\right)
$$

Proceeding as above this gives rise to the following divergent part of the four–point amplitude

$$
A^{(1)}_{\mathcal{M}}(p_1,p_2,p_3,p_4)\Big|_I = -i\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\,\frac{g^4}{(4\pi)^2}\,\epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot \epsilon_4\,\frac{4}{3}\,\frac{1}{\epsilon} \quad (3.45)
$$

Type II contributions come from the terms containing exponentials in (3.32, 3.33). From

(3.44) one gets

$$-\frac{i}{2}\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}))\frac{g_D^3}{(4\pi)^{\frac{D}{2}}}(2\alpha')^{\frac{4-D}{2}}\int_0^\infty d\tau\tau^{-\frac{D}{2}}e^{2\tau}\int_0^\tau d_{\nu_4}\int_0^{\nu_4}d_{\nu_3}\int_0^{\nu_3}d\nu_2\, e^{-2\alpha' m^2\tau}$$
$$\Big[32e^{-2\tau-2(\nu_2+\nu_3-\nu_4)}+32e^{-2\tau+2(\nu_2+\nu_3-\nu_4)}++32e^{2(\nu_2-\nu_3-\nu_4)}+32e^{-4\tau-2(\nu_2-\nu_3-\nu_4)}\Big]$$

whose evaluation leads to the divergent part

$$A^{(1)}_{\mathcal{M}}(p_1,p_2,p_3,p_4)\Big|_{II}=i\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\,\frac{g^4}{(4\pi)^2}\,\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4\,16\,\frac{1}{\epsilon} \tag{3.46}$$

Summing type I and type II we get

$$A^{(1)}_{\mathcal{M}}(p_1,p_2,p_3,p_4)\Big|_{I+II}=i\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\,\frac{g^4}{(4\pi)^2}\,\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4\,\frac{44}{3}\,\frac{1}{\epsilon} \tag{3.47}$$

Once again we obtain

$$A^{(1)}(p_1,p_2,p_3,p_4)\Big|_{\mathrm{I+II}}=-\frac{N-2}{2}\,\frac{g^2}{(4\pi)^2}\,\frac{11}{3}\,\frac{1}{\epsilon}A^{(0)}(p_1,p_2,p_3,p_4) \tag{3.48}$$

Eqs.(3.37, 3.43, 3.48) coincide with the results of the previous subsection. We remark that they are the one–loop quantum corrections expected in an $SO(N)$ gauge field theory in the background field formalism, [43, 42]. They correspond to a renormalization constant

$$Z_A=1+\frac{N-2}{2}\frac{g^2}{(4\pi)^2}\,\frac{11}{3}\,\frac{1}{\epsilon} \tag{3.49}$$

This amounts to one–loop renormalizability (in 4D) of the low energy effective action of the string theory with $so(N)$ Chan–Paton factors, that is the well–known fact that $SO(N)$ gauge field theory in 4D is renormalizable.

4. Field theory limit of gluon amplitudes with B field

Switching on a constant B field, on the basis of the discussion in section 2, amounts to replacing the propagator used in the previous section with the full propagator (2.12). Inserting it into the general formula (3.4) has a simple effect. The addition of the second term $-\frac{i}{2}\theta^{ij}\epsilon(\rho-\rho')$ does not affect derivatives of propagators, while it modifies the term $\prod_{r<s}e^{p_r G(\rho_r-\rho_s)p_s}$. This modification turns out to be very simple since the insertion points along the boundary of $\mathcal{M}$ are ordered, so that the relevant ϵ function is always either +1 or –1. As a consequence the corresponding exponential factors can be extracted from the moduli integral. In other words, the gluon amplitudes are multiplied by a global (noncommutative) factor

$$A^{(1)}(p_1,\dots,p_m)\to\prod_{r<s}e^{p_r\times p_s}A^{(1)}(p_1,\dots,p_m) \tag{4.1}$$

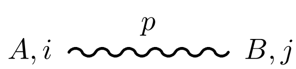
where $A^{(1)}(p_1,\dots,p_m)$ are the $B=0$ amplitudes and $p\times q=\frac{i}{2}p_i\theta^{ij}q_j$. The same is true also at tree level, [38, 34], and, on the basis of [22], it is likely to hold at any loop order, although we do not try to prove it here.

We can now infer that the analysis of the singularities in the field theory limit does not change with respect to the previous section, except for the global noncommutative factor in (4.1). We can therefore conclude that the structure of the divergent terms, as well as the renormalization constants, are the same as in the ordinary $SO(N)$ gauge theories. Therefore, if there exists a noncommutative gauge field theory that represents the low energy effective action of open strings with orthogonal CP factors in the presence of a constant B field, *this noncommutative gauge field theory must be one–loop renormalizable.*

5. Discussion

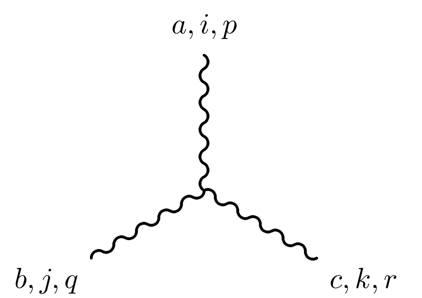
The above conclusion seems to imply that a renormalizable noncommutative gauge field theory with $so(N)$ Chan–Paton factors should exist. We recall that, even without resorting to an action, we can extract the gluon Feynman rules for this low energy field theory from the string tree amplitudes. They are as follows

gluon propagator.



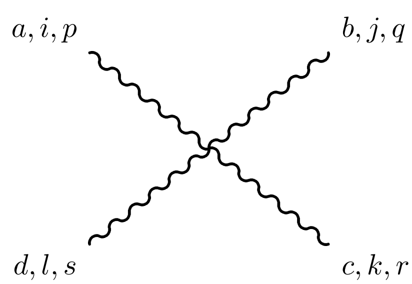
$$-\frac{i}{p^2}\delta_{ab}\hat{g}_{ij} \tag{5.1}$$

3–gluon vertex. The external gluons carry labels (a, i, p), (b, j, q) and (c, k, r) for the Lie algebra, momentum and Lorentz indices and are ordered in anticlockwise sense:



$$-gf^{abc}\cos(p \times q)\left(\hat{g}_{ij}\,(p-q)_k + \hat{g}_{jk}\,(q-r)_i + \hat{g}_{ki}(r-p)_j\right) \tag{5.2}$$

4–gluon vertex. The gluons carry labels (a, i, p), (b, j, q), (c, k, r) and (d, l, s) for Lie algebra, Lorentz index and momentum. They are clockwise ordered:



$$
\begin{aligned}
-ig^2 \quad & \Bigg\{ \Bigg[\; f^{xab} f^{xcd} \cos(p\times q)\cos(r\times s) \\
& - \left(4d^{abcd} - \frac{1}{3}(f^{xac}f^{xbd} + f^{xbc}f^{xad})\right) \sin(p\times q)\sin(r\times s)\Bigg] \, (\hat{g}_{ik}\hat{g}_{jl} - \hat{g}_{il}\hat{g}_{jk}) \\
& + \Bigg[f^{xac} f^{xdb} \cos(p\times r)\cos(s\times q) \\
& - \left(4d^{abcd} - \frac{1}{3}(f^{xcd}f^{xab} + f^{xcb}f^{xad})\right) \sin(p\times r)\sin(s\times q)\Bigg] \, (\hat{g}_{il}\hat{g}_{jk} - \hat{g}_{ij}\hat{g}_{kl}) \\
& + \Bigg[f^{xad} f^{xbc} \cos(p\times s)\cos(q\times r) \\
& - \left(4d^{abcd} - \frac{1}{3}(f^{xdb}f^{xac} + f^{xba}f^{xdc})\right) \sin(p\times s)\sin(q\times r)\Bigg] \, (\hat{g}_{ij}\hat{g}_{kl} - \hat{g}_{ik}\hat{g}_{jl}) \Bigg\}
\end{aligned}
\tag{5.3}
$$

We recall that this last vertex can be obtained from the string four–gluon amplitude only after subtracting two suitable tree one–particle reducible diagrams.

One can verify that the above Feynman diagrams can be obtained from the action suggested in [34]. From that action, which was called $NCSO(N)$, one can in addition extract the Feynman rules for the ghost fields. A natural question that arises is whether by applying these Feynman rules to compute one–loop amplitudes one gets the same results as the ones we obtained in the previous section. The surprising answer is that, if we apply Feynman rules in the ordinary way, we get a different result.

To illustrate the problem the simple $NCSO(2)$ case will do. From the string theory point of view it is rather easy to argue that the theory should not have UV divergences. Let us summarize our previous analysis. The one–loop contributions to open string amplitudes with $SO(N)$ Chan–Paton factors are of three types: planar (P) and nonplanar (NP) with the world–sheet of the annulus, and nonorientable (NO) with the world–sheet of the Möbius strip. Due to the structure of the string propagators on the annulus and on the Möbius strip, the contributions in the presence and in the absence of the B field for P and NO differ only by overall noncommutative factors. It follows that those contributions which become divergent in the field theory limit are the same whether B is there or not. Now in the ordinary $SO(N)$ case the divergent part comes from the planar contribution with a factor of N in front, and from the NO contribution with a factor of -2. So altogether the divergent field theory part is proportional to $N-2$, and therefore vanishes in the case $N=2$. This is obvious from the ordinary field theory side, because the theory is free. However, as we noticed above, this conclusion holds also in the noncommutative case. Therefore the $NCSO(2)$ theory should not give rise to UV divergences.

Now let us look at the one–loop order on the noncommutative field theory side. The Feynman rules are very simple in this case since only the four–point vertex is nonvanishing. Let us rewrite the four–gluon vertex adapted to this case

$$
\begin{aligned}
-2ig^2 \quad & [\; \cos(p\times r - q\times s)(\hat{g}_{ik}\hat{g}_{jl} + \hat{g}_{ij}\hat{g}_{kl} - 2\hat{g}_{il}\hat{g}_{jk}) \\
& + \cos(p\times s + q\times r)(\hat{g}_{il}\hat{g}_{jk} + \hat{g}_{ik}\hat{g}_{jl} - 2\hat{g}_{ij}\hat{g}_{kl}) \\
& + \cos(p\times s - q\times r)(\hat{g}_{ij}\hat{g}_{kl} + \hat{g}_{il}\hat{g}_{jk} - 2\hat{g}_{ik}\hat{g}_{jl}) \,]
\end{aligned}
\tag{5.4}
$$

The one–loop correction is infinite. So the theory needs a renormalization. What is worse is that the divergent part is not of the form (5.4), but

$$
\begin{aligned}
\sim \frac{g^4}{\epsilon} \quad & [\, \cos(p \times r - q \times s)(7\hat{g}_{ik}\hat{g}_{jl} + 7\hat{g}_{ij}\hat{g}_{kl} - 8\hat{g}_{il}\hat{g}_{jk}) \\
& + \cos(p \times s + q \times r)(7\hat{g}_{il}\hat{g}_{jk} + 7\hat{g}_{ik}\hat{g}_{jl} - 8\hat{g}_{ij}\hat{g}_{kl}) \\
& + \cos(p \times s - q \times r)(7\hat{g}_{ij}\hat{g}_{kl} + 7\hat{g}_{il}\hat{g}_{jk} - 8\hat{g}_{ik}\hat{g}_{jl})\,]
\end{aligned}
\tag{5.5}
$$

In order to eliminate this divergence we need a counterterm of the form

$$
\sim (7A_i * A^i * A_j * A^j - 4A_i * A_j * A^i * A^j) \tag{5.6}
$$

Therefore not only the $NCSO(2)$ gauge field theory is not finite, but the divergent part breaks the gauge symmetry. One might argue that $NCSO(N)$ gauge theories are nonlocal theories and it is perhaps too much hoping for another miracle like the renormalizability of noncommutative $U(N)$ theories to happen also in this case. However the fact the string theory with $so(N)$ CP factors in the presence of a B field is well–behaved and its field theory limit is well–defined, suggests another possible solution to the puzzle. After a moment's thought one realizes that the element where field theory and string theory diverge is not the Feynman rules themselves (or the action they come from) but their application in the one–loop calculation. We have applied them in the usual way, but apparently that is (partially) wrong. The possibility we envisage here is that there is a deformation not only in space-time but also in the Lie algebra 'direction'. Taking into account this deformation should lead to one–loop renormalizability. We will elaborate on this idea in a forthcoming paper.

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A. $SO(N)$ tensors

In this Appendix we collect the conventions relevant for the $so(N)$ Lie algebra tensors and traces. We denote the Lie algebra generators by t^a, where $a = 1, \ldots, \frac{N(N-1)}{2}$. They are real antisymmetric matrices with Lie bracket and normalization defined by

$$
[t^a, t^b] = f^{abc} t^c, \qquad \mathrm{tr}(t^a t^b) = -\frac{1}{2}\delta^{ab} \tag{A.1}
$$

tr is the trace in the fundamental representation and summation over repeated indices is understood. With these conventions we find

$$
\mathrm{tr}(t^a t^b t^c) = -\frac{1}{4} f^{abc}
$$

Unlike the $u(N)$ Lie algebra, $so(N)$ does not possess a third order invariant symmetric tensor. The fourth order invariant symmetric tensor is defined by means of

$$\mathrm{Sym}(t^a t^b t^c) \equiv \frac{1}{6}\left(t^a t^b t^c + 5 \text{ permutations}\right) \equiv d^{abcd} t^d \tag{A.2}$$

We find

$$\mathrm{tr}(t^a t^b t^c t^d) = -\frac{1}{2} d^{abcd} - \frac{1}{6} f^{abx} f^{xcd} + \frac{1}{12} f^{xac} f^{xbd} \tag{A.3}$$

Evaluating one–loop Feynman diagrams in field theory requires the corresponding traces in the adjoint representation. Let us denote by F^a the matrices

$$(F^a)_{bc} = f^{abc}$$

and by Tr the traces in the vector space of the adjoint representation. Then one finds

$$\mathrm{Tr}(F^a F^b) = -\frac{1}{2}(N-2)\delta^{ab} \ , \qquad \mathrm{Tr}(F^a F^b F^c) = \frac{1}{4}(N-2) f^{abc} \tag{A.4}$$

(A.5)

and

$$\begin{aligned} \mathrm{Tr}(F^a F^b F^c F^d) \ &= \ -\frac{N-2}{2} d^{abcd} + \frac{1}{4}\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right) \\ &\quad + \frac{N-2}{12}\left(f^{adx} f^{xbc} - f^{abx} f^{xcd}\right) \end{aligned} \tag{A.6}$$

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